ELECTRICAL SIMULATION OF BOUNDARY CONDITIONS IN NONSTEADY PROBLEMS OF HEAT AND MASS TRANSFER

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A method of modeling heat- and mass-transfer problems with boundary conditions of the second and third kinds is described. Simplified model circuits are presented.

1. We consider the system of differential transport equations given in dimensionless parameters [1, 2],

$$\frac{\partial T}{\partial \mathrm{Fo}} = \nabla^2 T + \mathrm{Ko}^* \frac{\partial \Theta}{\partial \mathrm{Fo}}, \qquad \frac{\partial \Theta}{\partial \mathrm{Fo}} = \mathrm{Lu}\,\nabla^2 \Theta + \mathrm{Lu}\,\mathrm{Pn}\,\nabla^2 T. \tag{1}$$

We assume that the normal heat and mass fluxes at any point of the surface and any moment of time have been determined, i.e., the Kirpichev numbers $\mathrm{Ki}_q(X_s; Y_s; Z_s; \mathrm{Fo})$ and $\mathrm{Ki}_m(X_s; Y_s; Z_s; \mathrm{Fo})$ are given. In this case, the boundary conditions of the second kind take the form

$$\frac{\partial T}{\partial N} = -\mathrm{Ki}_q, \qquad \frac{\partial \Theta}{\partial N} + \mathrm{Pn}\frac{\partial T}{\partial N} = -\mathrm{Ki}_m, \tag{2}$$

whence the normal derivatives of T and Θ at the surface

$$\left(\frac{\partial T}{\partial N}\right)_s = -\mathrm{Ki}_q; \quad \left(\frac{\partial \Theta}{\partial N}\right)_s = -\mathrm{Pn}\,\mathrm{Ki}_q - \mathrm{Ki}_m. \tag{3}$$

This means that if we construct an electronic model with two RC networks [3], the electric currents i_1 and i_2 will be given at the boundaries of these networks. These currents may be expressed as follows:

$$\bar{i}_1 = -\frac{(\mathrm{grad}\, v)_n}{R_1} = \frac{v^*}{R_1 l}\mathrm{Ki}_q \bar{N}, \qquad \bar{i}_2 = -\frac{(\mathrm{grad}\, w)_n}{R_2} = \frac{w^*}{R_2 l}(\mathrm{Pn}\,\mathrm{Ki}_q + \mathrm{Ki}_m)\bar{N}, \tag{4}$$

where $N = \bar{n}/l$ is the dimensionless outward normal. (Here, $\bar{n}$ is the normal to the surface of the simulated object.)

The currents can be created by applying high voltages v_c and w_c from a potentiometer across the high resistances R_{b_1} and R_{b_2}:

$$i_1 = \frac{v_c - v_s}{R_{b_1}}, \quad i_2 = \frac{w_c - w_s}{R_{b_2}}.$$

Since $v_c \gg v_s$ and $w_c \gg w_s$, $i_1 = v_c/R_{b_1}$, $i_2 = w_c/R_{b_2}$. If Ki_q and Ki_m depend on Fo, the voltages v_c and w_c must similarly depend on τ_e.

If the effect of the temperatures and mass-transfer potentials at certain points of the body on the flows through the surface j_k and j_m is known, then by means of a system of amplifiers it is possible automatically to change the values of the voltages v_c and w_c, and thereby create the necessary i_1 and i_2 at the boundary of the model.

If we create electrical models of the flows and the boundary layer for a porous body in which the electrical currents i_1 and $i_2 + (C/(C_1 + C))i_1$ are analogs of j_q and j_m, it follows that by the simple contact of conductors with the surface of the model we can realize the boundary conditions of the fourth kind (see [4]):

$$T_{s_1} = T_{s_2}; \; \Theta_{s_1} = \Theta_{s_2}. \qquad j_{q_1} = j_{q_2}; \; j_{m_1} = j_{m_2}.$$

2. We consider boundary conditions of the third kind, when the coefficients of heat and mass transfer α_q and α_m vary only slightly over a certain period of time and can be assumed constant:

$$\left(\frac{\partial T}{\partial N}\right)_s - (1-\varepsilon)\,\mathrm{Ko}\,\mathrm{Lu}\,\mathrm{Bi}\,(\Theta_c - \Theta_s) = \mathrm{Bi}_q(T_c - T_s), \qquad \left(\frac{\partial \Theta}{\partial N}\right)_s + \mathrm{Pn}\left(\frac{\partial T}{\partial N}\right)_s = \mathrm{Bi}_m(\Theta_c - \Theta_s). \tag{5}$$

We first construct a model of the boundary conditions for the special case

$$\left(\frac{\partial T}{\partial N}\right)_s = \mathrm{Bi}_q(T_c - T_s), \qquad \left(\frac{\partial \Theta}{\partial N}\right)_s = \mathrm{Bi}_m(\Theta_c - \Theta_s). \tag{6}$$

Here, the heat flow through the surface $(\partial T/\partial N)_s$ is determined only by the difference between the ambient and surface temperatures.

We connect to points on the surface of the electrical model the boundary resistances R_{b_1} for network I and R_{b_2} for network III (the one-dimensional case is represented in Fig. 1). On the basis of Kirchhoff's law, we have

$$-\frac{1}{R_1}\frac{\partial v}{\partial n_e} = \frac{v_s - v_c}{R_{b_1}}; \quad -\frac{1}{R_2}\frac{\partial w}{\partial n_e} = \frac{w_s - w_c}{R_{b_2}}. \tag{6a}$$

We write (6a) in dimensionless coordinates, introducing the Biot numbers

$$\mathrm{Bi}_q = R_1 l_e / R_{b_1} \text{ and } \mathrm{Bi}_m = R_2 l_e / R_{b_2}.$$

In these criteria, the specific resistances R_1 and R_2 (Mohm · m) are referred to unit volume of the conducting medium and Rb_1 and Rb_2 (Mohm · m^2) are referred to unit surface. Also introducing the dimensionless potentials $V = v/v^*$, $W = w/w^*$ and the dimensionless normal, we write the boundary conditions in the form

$$\frac{\partial V}{\partial N} + \mathrm{Bi}_q (V_s - V_c) = 0; \quad \frac{\partial W}{\partial N} + \mathrm{Bi}_m (W_s - W_c) = 0.$$

These conditions should coincide with the boundary conditions (6) of the simulated object, for which the equations

$$\mathrm{Bi}_q = \frac{\alpha_q l}{\lambda_q} = \frac{R_1 l_e}{R_{b_1}}; \quad \mathrm{Bi}_m = \frac{\alpha_m l}{\lambda_m} = \frac{R_2 l_e}{R_{b_2}} \tag{7}$$

must be satisfied.

We now turn to the general case of electrical simulation of the boundary conditions (5). It is not possible to proceed by constructing a model of these conditions by connecting certain resistances with a constant potential at the ends, as we did with the boundary conditions (6). However, we can proceed by simulating not the potentials T and Θ but the functions φ and ψ in the linear combinations

$$\begin{aligned} T &= a\varphi + b\psi + T_c, \\ \Theta &= \varphi + \psi + \Theta_c, \end{aligned} \tag{8}$$

where a and b are certain constant coefficients.

This transformation does not change the form of the starting differential equations (1). If, however, we substitute (8) into the boundary conditions (5), setting

$$\Big[a,\ b = \big[\mathrm{Bi}_m - \mathrm{Bi}_q - (1-\varepsilon)\,\mathrm{Ko\,Lu\,Bi}_m\,\mathrm{Pn} \pm \Big] \times \\ \times \pm [\mathrm{Bi}_m - \mathrm{Bi}_q - (1-\varepsilon)\,\mathrm{Ko\,Lu\,Bi}_m\,\mathrm{Pn}]^2 - \\ - 4(1-\varepsilon)\,\mathrm{Ko\,Lu\,Bi}_q\,\mathrm{Bi}_m\,\big]^{1/2} \times (2\,\mathrm{Pn\,Bi}_q)^{-1}, \tag{8a}$$

(the plus sign corresponds to a, the minus sign to b), it is seen that the boundary conditions for φ and ψ take a form analogous to conditions (6), and they can be simulated by connecting resistances Rb_1 and Rb_2 at the boundary. For example, in the one-dimensional case, the model is represented by the circuit in Fig. 1.

Substituting (8) into (1), we obtain the dimensionless differential equations in φ and ψ,

$$\begin{aligned} \frac{\partial \varphi}{\partial A_1} &= \nabla^2 \varphi + A_2 \frac{\partial \psi}{\partial A_1}, \\ \frac{\partial \psi}{\partial A_1} &= A_3 \nabla^2 \psi + A_4 \frac{\partial \varphi}{\partial A_1}, \end{aligned} \tag{9}$$

where the new criteria A_2, A_3, and A_4 are expressed in terms of a, b, Lu, Pn, Ko*, and the criterion A_1 is proportional to Fo.

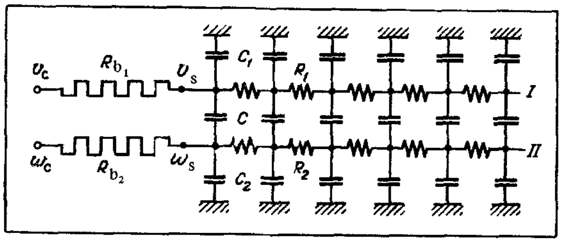

Fig. 1. Electrical model for simulating the boundary value problem of heat and mass transfer.

Thus, the problem has been reduced to the solution of Eqs. (9) with boundary conditions analogous to (6). The electrical parameters R_1, R_2, C_1, C_2, C should be selected so that

$$A_1 = \frac{\tau_e}{R_1 (C_1 + C)\, l_e^2}; \quad A_2 = \frac{Cw^*}{(C_1 + C)\, v^*};$$
$$A_3 = \frac{R_1 (C_1 + C)}{R_2 (C_2 + C)}, \quad A_4 = \frac{Cv^*}{(C_2 + C)\, w^*}.$$

In conducting the experiment, there is no need to measure separately the auxiliary variables φ and ψ and then calculate T and Θ from Eqs. (8). It is simpler to assemble a measuring circuit in which quantities proportional to φ and ψ are summed and thus measure the quantities $T - T_c$ and $\Theta - \Theta_c$ directly.

Obviously, by this means one can find a and b not for any similarity criteria Bi_q, Bi_m, Lu, Pn, Ko and ε, but only for those for which the radicand in (8a) is positive.

3. The criteria Lu, Pn, and Ko* depend only on the thermophysical quantities, and, in the model, on R_1, R_2, C, C_1, C_2, v^* and w^* [3]. To simplify the circuit, some of the electrical parameters can be set equal to zero.

We can take: a) $C_1 = 0$; or b) $C_2 = 0$. (C_1 and C_2 cannot be simultaneously equal to zero, since, in this case, the denominator of the Lu number would vanish.) For these two special cases, a) and b), we have two corresponding simplified model circuits, which are shown for the one-dimensional problem in Fig. 2. The similarity criteria take the form

$$\text{a)}\ \mathrm{Lu} = \frac{R_1 C}{R_2 C_2}, \quad \text{b)}\ \mathrm{Lu} = \frac{R_1 (C_1 + C)^2}{R_2 C C_1},$$
$$\mathrm{Ko}^* = \frac{w^*}{v^*}, \qquad \mathrm{Ko}^* = \frac{w^* C}{v^* (C_1 + C)},$$
$$\mathrm{Pn} = \frac{R_2 v^*}{R_1 w^*}, \qquad \mathrm{Pn} = \frac{R_2 C v^*}{R_1 (C_1 + C) w^*}.$$

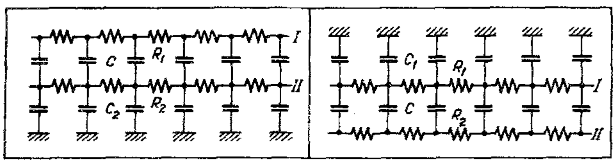

Fig. 2. Special cases of electrical circuits for simulating problems of heat and mass transfer.

Hence, it is clear that, in circuit a), the Ko* number does not depend on the distributed electrical parameters but only on the scales of the electrical quantities v* and w*. Therefore, if Ko* is very small, then, in conducting the experiment, the voltage w should be amplified before the measurements are made. In circuit a), the physical significance of the Lu number is particularly apparent: this criterion is equal to the ratio of the time constants of the networks R_1C and R R_2C_2.

Knowing the ratio of the specific resistances and capacitances,

$$\frac{R_2}{R_1} = \text{Pn}\,\text{Ko}^* \text{ and } \frac{C}{C_2} = \text{Lu}\,\text{Pn}\,\text{Ko}^*,$$

we can select suitable time constants of the two networks R_1C and $R_2(C + C_2)$.

The dimensionless heat and mass fluxes are, respectively, equal to

$$\bar{J}_q = \frac{\bar{j}_q l}{\lambda_q t^*} = \frac{\bar{i}_1 R_1 l_e}{v^*}; \quad \bar{J}_m = \frac{\bar{j}_m l}{\lambda_m \Theta^*} = \frac{R_2 l_e}{w^*}\,(\bar{i}_1 + \bar{i}_2),$$

i.e., the electrical current i_1 simulates both the heat flux and the mass flux due to thermal diffusion.

4. It is clear from Fig. 1 that the circuit is symmetrical, in the sense that it is possible to exchange the roles of networks I and II, and if we construct an electrical model for certain values of the similarity criteria Fo_1, Lu_1, Pn_1 and Ko_1^*, we find that the same model can also serve for simulating some other problem with similarity criteria Fo_2, Lu_2, Pn_2 and Ko_2^*.

Equations (1) can be written in the form

$$\frac{\partial T_1}{\partial \text{Fo}_1} = (1 + \text{Ko}_1^* \text{Lu}_1 \text{Pn}_1)\,\nabla^2 T_1 + \text{Lu}_1 \text{Ko}_1^* \nabla^2 \Theta_1,$$

$$\frac{\partial \Theta_1}{\partial \text{Fo}_1} = \text{Lu}_1 \text{Pn}_1 \nabla^2 T_1 + \text{Lu}_1 \nabla^2 \Theta_1.$$

The equations for the similarity criteria with subscript 2 are written analogously.

We exchange the roles of T_1 and Θ_2; Θ_1 and T_2. Then, the indicated pairs of equations will coincide identically, if the coefficients

$$\text{Fo}_2 \text{Lu}_2 = (1 + \text{Lu}_1 \text{Pn}_1 \text{Ko}_1^*)\,\text{Fo}_1; \quad \text{Fo}_2 \text{Lu}_2 \text{Pn}_2 = \text{Lu}_1 \text{Ko}_1^* \text{Fo}_1;$$

$$\text{Fo}_2 \text{Lu}_2 \text{Ko}_2^* = \text{Lu}_1 \text{Pn}_1 \text{Fo}_1; \quad \text{Fo}_2 (1 + \text{Lu}_2 \text{Ko}_2^* \text{Pn}_2) = \text{Lu}_1 \text{Fo}_1.$$

coincide.

We introduce the similarity criterion used in analytic calculations [1]:

$$\xi = \frac{1}{\text{Lu}} + \text{Pn}\,\text{Ko}^*.$$

Then,

$$\text{Lu}_2 = \text{Lu}_1 \xi_1^2; \quad \text{Ko}_2^* = \text{Pn}_1 \frac{1}{\xi_1};$$

$$\text{Pn}_2 = \text{Ko}_1^* \frac{1}{\xi_1}; \quad \text{Fo}_2 = \text{Fo}_1 \frac{1}{\xi_1}. \tag{10}$$

We consider the analogous duality of the problem for electrical models (see Fig. 1), in this case, the roles of networks I and II are interchangeable. In one case,

$$\text{Fo}_1 = \frac{\tau_e}{R_1 (C_1 + C)\, l_e^2}; \quad \text{Ko}_1^* = \frac{C}{(C_1 + C)} \frac{w^*}{v^*}; \ \text{Pn}_1 = \ldots$$

In the other,

$$\text{Fo}_2 = \frac{\tau_e}{R_2 (C_2 + C)\, l_e^2}; \ \text{Ko}_2^* = \ldots$$

In electrical parameters, the criterion ξ takes the form

$$\xi = \frac{R_2 (C_2 + C)}{R_1 (C_1 + C)}$$

and all of relations (10) remain in force. The criterion ξ is equal to the ratio of the time constants for networks I and II. The RC network constructed for simulating the problem is suitable for modeling a field not only with given Lu number but also with Lu ξ^2 in the dual problem. By varying the voltages v* and w*, it is possible to vary the Pn and Ko* numbers, but only in such a way that their product Fe = Pn Ko* remains constant. The electrical model of heat and mass transfer is characterized by two similarity criteria, for example, Lu and ξ.

NOTATION

t is the temperature, °C; θ is the mass transfer potential, °M; λ_q and λ_m are the thermal and mass conductivities, respectively; ε is the ratio of the change of mass due to phase transformation to the total change of mass; l is the characteristic linear dimension; Fo is the Fourier number; Lu is the Luikov number; Pn is the Posnov number; Ko* is the modified Kossovich number; t* and θ* are the characteristic temperatures and mass transfer potential; τ is the time; v* and w* are certain specific potential differences; C_1, C_2, and C are in $\mu F/m^3$; R_1 and R_2 are the specific resistances and capacitances of the system of conducting media, Mohm · m; τ_e is the model time, sec; R_{b_1}, R_{b_2} are the boundary resistances, Mohm · m^2; n_e is the normal to the model surface; T = t/t*; $\Theta = \theta/\theta^*$. The subscript s relates to a surface point; c corresponds to a parameter of the medium.

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